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دراسة المشتقة الكسرية من رتبة متغيرة لدوال القوى ذات رتبة كسرية

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الملخص

في هذا البحث سيتم تقديم دراسة لخصائص المشتقة الكسرية لدوال القوى عندما تكون رتبة التفاضل دالة في متغير مختلف عن المتغير المستقل لدالة القوى وتحديد كيفية تأثير رتبة التفاضل المتغيرة على شكل المشتقة.

الكلمات المفتاحية: المشتقة الكسرية، المشتقة الكسرية من رتبة متغيرة، دالة القوى , دالة جاما , مشتقة كابوتو

A Study of Fractional Variable Order Derivatives of Fractional Power Functions

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Abstract

In this research, we will present a study on the properties of fractional derivatives of power functions when the order of differentiation is a function of a different variable. We aim to determine how the variable order of differentiation affects the form of the derivative.

Keywords: , Caputo's derivative, Gamma function, Fractional derivative, Fractional derivative of variable order and Power function.

Introduction

Since the concept of generalizing ordinary derivatives from integer orders to fractional and fractional derivatives emerged in 1968 (Oldham& Spanier, 1974), studies and research in this field have not ceased. This generalization is significant in many scientific areas, such as:

Physics: Where fractional derivatives are used to describe dynamic and complex phenomena (Oldham& Spanier, 1974).

Engineering: In system design and structural analysis. (Kakhki, et.al 2010)..

Medicine: In modeling biological processes and chemical interactions (Machado & Bates¸2017).

Economics: To analyze complex economic systems (Luo, et.al, 2018).

Computer Science: In signal and image processing (Arora, et.al, 2022)
Both Samko and Ross (1993) presented a more extensive generalization of the fractional derivative. This generalization is the fractional derivative of variable order, where the order of the derivative is a function that may depend on the same spatial variable, the time variable, or another variable altogether. (Samko& Ross, 1993)
Once this generalization emerged, studies in this field did not cease. Most definitions of fractional derivatives, such as Riemann-Liouville and Caputo, were generalized to the concept of fractional differentiation of variable order.
The fractional differentiation of variable order has been used in many important applications, such as:
Modeling Complex Systems: In physics and engineering to describe systems with memory and hereditary properties (Coimbra, 2003).
Control Theory: For designing controllers in dynamic systems with fractional dynamics (Patnaik, et.al, 2020).
Signal Processing: In filtering and analyzing signals with non-integer order derivatives. (Ortigueira, 2006).
Biological Systems: To model processes like diffusion and population dynamics (Jafari, et.al, 2021)
Finance: In modeling stock prices and financial derivatives where traditional models fall short (Baz & Chacko, 2004)

The fractional derivative of variable order using the Caputo concept:

The primary advantage of the Caputo concept is that the initial and boundary conditions for differential equations using the Caputo fractional derivative are similar to those for integer-order differential equations. This means that they can be derived in the same way (Sikora, 2020).
Caputo's definition of the fractional derivative of variable order can be written as:

$$D^{\alpha(t)}f(x)=\frac{1}{\Gamma(n-\alpha(t))}\int_a^x \frac{f^{(n)}(s)ds}{(x-s)^{\alpha(t)-n+1}} \qquad (1)$$

Where $n-1<\alpha(t)<n$ (Sweilam & Almrawm, 2011).
Therefore, when finding a derivative of a variable order, it is important to ensure that the function of the order is bounded. Note that the derivative here will be a function of two variables: the main variable of the function and the variable of the order function.

Some properties of Caputo's derivative of variable order.

The derivative of variable order using the Caputo definition has algebraic properties that are similar to those of the fractional formula such as:

(i) The linearity:

$$D^{\alpha(t)}\big(af(x)+bg(x)\big)=aD^{\alpha(t)}f(x)+bD^{\alpha(t)}g(x)$$

Where a and b are constant, and f and g are continuous functions.

(ii) The Leibniz rule:

$$D^{\alpha(t)}\big(f(x)g(x)\big)=\sum_{k=0}^{\infty}f^{(k)}(x)D^{\alpha(t)-k}g(x)$$

Where f(x) and g(x) along with all its derivatives are continuous.

(iii)

$D^{\alpha(t)}\big(D^m f(x)\big)=D^{\alpha(t)+m}f(x),\ (m=0,1,2,\dots;n-1<\propto(t)<n).$

For more properties see (Podlubny, 1999), (Oldham& Spanier, 1974)

3. Fractional variable order derivatives of the power function.

In this section the fractional variable order derivative of power functions are obtained as in the following theorem (Garrappa, et.al, 2019). The formula of the power function that we study is:

$$f(x)=(x-a)^{v}$$

Where a, and v are real values.

Theorem

If v and a are real values, such that $v>-1$ then

$$D^{\propto(t)}(x-a)^{v}=\begin{cases}0 & v\in\{0,1,\dots,n-1\}\\ \dfrac{\Gamma(v+1)}{\Gamma(v-\alpha+1)}(x-a)^{v-\propto(t)} & v>n-1\\ non\ existing & o.w\end{cases}$$

Where $n-1<\propto(t)<n$.

The proof:

$$D^{\alpha(t)}(x-a)^{v}=\frac{1}{\Gamma(n-\alpha(t))}\int_a^x\frac{f^{(n)}(s-a)^{v}}{(x-s)^{\alpha(t)-n+1}}ds$$

We know that, when n is integer value:

$$f^{(n)}(t-a)^{v}=v(v-1)\dots(v-n+1)(t-a)^{v-n}$$
$$=\frac{\Gamma(v+1)}{\Gamma(v-n+1)}(t-a)^{v-n}$$

And, in the fractional case:

$$D^{\alpha}(x-a)^{v}=\frac{\Gamma(v+1)}{\Gamma(n-\alpha)\Gamma(v-n+1)}\int_a^x\frac{(t-a)^{v-n}}{(x-t)^{\alpha-n+1}}dt$$

So,

$$D^{\alpha(t)}(x-a)^{v}=\frac{\Gamma(v+1)}{\Gamma(n-\alpha(t))\Gamma(v-n+1)}\int_{a}^{x}\frac{(s-a)^{v-n}}{(x-s)^{\alpha(t)-n+1}}ds$$

Let $s=a+z(x-a)$, then
When $s=a$, z=0, and if $s=x,\ z=1$.
Hence,

$$(x-s)^{\alpha(t)-n+1}=(x-a)^{\alpha(t)-n+1}(1-z)^{\alpha(t)-n+1}$$

And

$$(s-a)^{v-n}=z^{v-n}(x-a)^{v-n}$$

Then

$$D^{\alpha(t)}(x-a)^{v}=\frac{\Gamma(v+1)}{\Gamma\big(n-\alpha(t)\big)\Gamma(v-n+1)}\int_{0}^{1}\frac{z^{v-n}(x-a)^{v-n}}{(x-a)^{\alpha(t)-n+1}(1-z)^{\alpha(t)-n+1}}\,.(x-a)dz$$

$$=\frac{\Gamma(v+1)(x-a)^{v-\alpha(t)}}{\Gamma\big(v-\alpha(t)\big)\Gamma(v-n+1)}\,.\int_{0}^{1}z^{v-n}\,(1-z)^{n-\alpha(t)-1}dz$$

$$=\frac{\Gamma(v+1)(x-a)^{v-\alpha(t)}}{\Gamma(v-\alpha(t)+1)\Gamma(v-n+1)}\,.B(v-n+1,n-\alpha(t))$$

$$=\frac{\Gamma(v+1)}{\Gamma(v-\alpha(t)+1)}(x-a)^{v-\alpha(t)}.\quad(2)$$

As a special case
If $\propto(t)>0\,,and\ n-1<\propto(t)<n$. then for any $k\in\mathbb{N}$:

$$D^{\propto(t)}x^{k}=\begin{cases}0 & \text{if } k<\propto(t)\\ \displaystyle\sum_{l=n}^{k}\frac{k!}{(k-l)!\,(l-\propto(t)+1)}x^{l-\propto(t)} & o.w\end{cases}\quad(3)$$

Challenges in Determining the Gamma Function for Variable Order Derivatives

It's important to remember that finding the fractional derivative of variable order involves computing the value of the Gamma function for the order of the derivative, which is a challenging task for researchers in this field.
This problem can be resolve by using one of the following methods:

Using Gamma Function Approximations

- Approximation Using Series: Taylor series or other series can be used to approximate the Gamma function at specific points (Fitzpatrick, 2000).

- Numerical Methods: Methods such as the Trapezoidal rule or Simpson's rule can be used to estimate the value of the Gamma function (Gil, 2007).

Numerical Analysis

- Numerical Techniques: Include techniques such as Newton-Raphson algorithms or iterative methods that can be used to compute fractional derivatives (Press, et..al, 1992).
- Special Functions: Some software provides special functions for numerically calculating fractional derivatives. (Gil, et.al, 2007).

Symmetry Methods

- Asymptotic Analysis: Using techniques such as symmetry in functions or derivatives to obtain approximate results.(Min, et.la, 2023)
- Transformations: Some mathematical transformations may help simplify calculation (Andrews, et.al, 1999).

All the previous points are potential topics for future research papers. However, in this study, we will present an illustration of the fractional derivative of variable order by calculating it point wise within the domain of the order function, and then using MATLAB to plot this derivative.

Analysis of Non-negativity and Boundedness of the Order Function

In the context of fractional calculus, it is essential to analyze the properties of the order function to ensure the existence of the fractional variable order derivative. Two critical conditions must be satisfied: non-negativity and boundedness.

Non-negativity: The order function must be non-negative, meaning that its values should be greater than or equal to zero. This condition is crucial because negative values could lead to undefined behavior in the calculation of the Gamma function.

Boundedness: Additionally, the order function must be bounded, specifically less than the order of the power function involved. This restriction ensures that the Gamma function can be computed effectively, preventing potential divergences or singularities in the calculations. These conditions are necessary for the proper definition and computation of the fractional derivative of variable order, allowing researchers to explore its applications in various fields effectively.

6. The examples

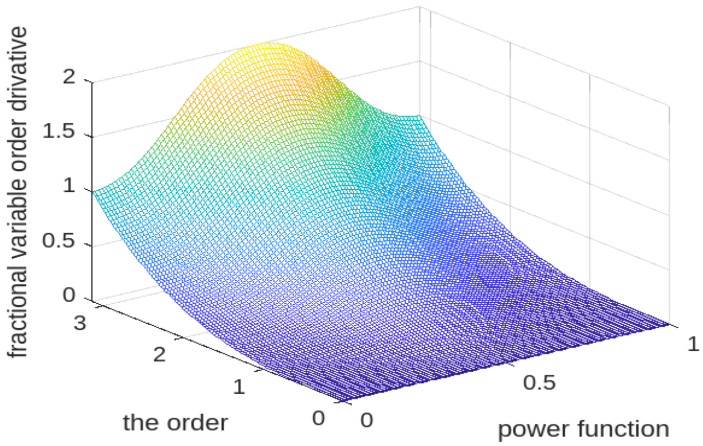
Example (1): in this example we use Caputo's definition to fine the fractional of variable order to the function$f(x) = x^2$, when the order is the function $\propto (t) = sin^2(t)$.

By using (3), and as is known that $0 \leq sin^2(t) \leq 1$, then

$$\mathrm{D}^{sin^2(t)}\mathrm{x}^2 = \frac{2!}{(2-\mathrm{l})!\,\Gamma(\mathrm{l}-sin^2(t)+1)}\ \mathrm{x}^{\mathrm{l}-sin^2(t)}$$

$$= \frac{2}{\Gamma(2 - sin^2(t))} \, x^{1-sin^2(t)}$$

The figure (1) shows the graph of the derivative of fractional variable order. As expected, the graph of the derivative transitions from the quadratic function, which is obtained when the order is equal to zero, to a linear form that shows its



maximum value of one.

Figure. 1

Example 2: The current example relates to finding the derivative of a power function $(x) = (x-2)^{3.5}$, when the order of the derivative is the function $\alpha(t) = 2 - cos^2(t)$.

In this example, to ensure the convergence of the solution and to make it defined in the set of real numbers, we will assume that the values of x are greater than 2, for more details, see (Almrawm & Ajaib, 2024)

By using (2), we have

$$D^{2-cos^2(t)}(x-2)^{3.5} = \frac{\Gamma(3.5+1)}{\Gamma(3.5 - 2 + cos^2(t) + 1)}$$

$$.\,(x-2)^{3.5-2+cos^2(t)}.$$

$$= \frac{\Gamma(4.5)}{\Gamma(2.5 + cos^2(t))}(x-2)^{1.5+cos^2(t)}$$

The figure (2) shows the graph of the derivative of fractional variable order that was studied in this example..

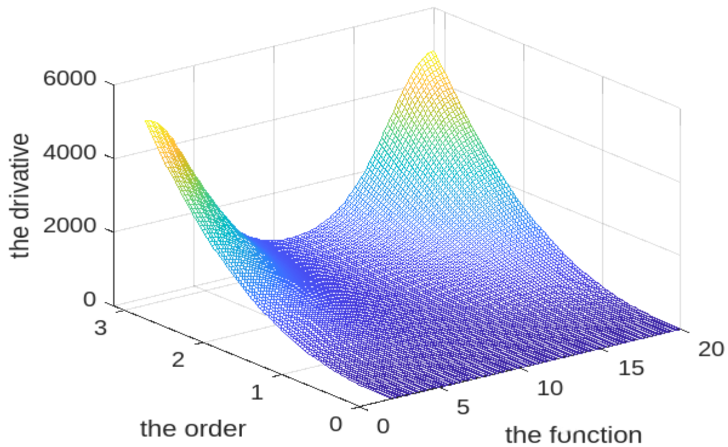


Figure. 2

Conclusion:

Variable order fractional derivatives have recently gained attention from researchers due to their significance in various applications. This study focuses on variable order fractional derivatives of power functions. We treat the order function as a variable separate from the power function variable, which can represent time or any other variable that may affect the order of the derivative. We address some challenges that researchers may encounter in this field and propose possible solutions. Additionally, we illustrate with examples and graphs how the fractional derivative changes with variations in the order function**.**

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